

SEMILOCAL CONVERGENCE OF TWO ITERATIVE METHODS FOR SIMULTANEOUS COMPUTATION OF POLYNOMIAL ZEROS*

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Abstract

In this paper we study some iterative methods for simultaneous approximation of polynomial zeros. We give new semilocal convergence theorems with error bounds for Ehrlich's and Norein's iterations. Our theorems generalize and improve recent results of ZHENG and HUANG [J. Comput. Math. 18 (2000), 113–122], PETKOVIĆ and HERCEG [J. Comput. Appl. Math. 136 (2001), 283–307] and NEDIĆ [Novi Sad J. Math. 31 (2001), 103–111]. We also present a new sufficient condition for simple zeros of a polynomial.

Key words: polynomial zeros, simultaneous methods, semilocal convergence, Ehrlich method, Börsch-Supan method, Norein method

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1 Introduction and notations

Let f be a monic complex polynomial of degree $n \geq 2$. A point ξ in \mathbb{C}^n is said to be a root-vector of f if its components are exactly the zeros of f with their multiplicities. There are a lot of iterations for simultaneous computation of all zeros of f (see the monographs of SENDOV, ANDREEV, KJURKCHIEV [1], PETKOVIĆ, HERCEG, ILIC [2] and KYURKCHIEV [3]). The famous one is WEIERSTRASS' iteration [4]

$$(1) \quad z^{k+1} = z^k - W(z^k), \quad k = 0, 1, 2, \dots,$$

where the operator W in \mathbb{C}^n is defined by $W(z) = (W_1(z), \dots, W_n(z))$ with

$$W_i(z) = \frac{f(z_i)}{\prod_{j \neq i} (z_i - z_j)} \quad (i = 1, 2, \dots, n).$$

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Another iteration for simultaneous finding all zeros of f is EHRlich's iteration [5]

$$(2) \quad z^{k+1} = F(z^k), \quad k = 0, 1, 2, \dots,$$

where the operator F in \mathbb{C}^n is defined by $F(z) = (F_1(z), \dots, F_n(z))$ with

$$(3) \quad F_i(z) = z_i - \frac{f(z_i)}{f'(z_i) - f(z_i) \sum_{j \neq i} 1/(z_i - z_j)}.$$

WERNER [6] has proved that the iteration function F can also be written in the form

$$(4) \quad F_i(z) = z_i - \frac{W_i(z)}{1 + \sum_{j \neq i} W_j(z)/(z_i - z_j)}.$$

Ehrlich's method (2) with the iteration function F defined by (4) instead of (3) is known as Börsch-Supan's method since in such form it was proposed for the first time by BÖRSCH-SUPAN [7]. The following iteration is due to NOUREIN [8]

$$(5) \quad z^{k+1} = G(z^k), \quad k = 0, 1, 2, \dots,$$

where the operator G in \mathbb{C}^n is defined by $G(z) = (G_1(z), \dots, G_n(z))$ with

$$(6) \quad G_i(z) = z_i - \frac{W_i(z)}{1 + \sum_{j \neq i} W_j(z)/(z_i - z_j - W_i(z))}.$$

Nourein's method is also known as Börsch-Supan method with Weierstrass' correction.

Since 1996, some authors [9, 10, 11, 12, 13, 14, 15, 16] have obtained semilocal convergence theorems for Ehrlich's and Nourein's methods from data at one point. The best results on Ehrlich's method are due to ZHENG and HUANG [14] and PETKOVIĆ and HERCEG [15]. The best results on Nourein's method are due to ZHENG and HUANG [14] and NEDIĆ [16].

In this paper, we present new semilocal convergence theorems for Ehrlich's and Nourein's iterations which generalize and improve all previous results in this area. We also present a new sufficient condition for simple zeros of a polynomial. The main results of the paper (Theorems 2.1, 3.1 and 4.1) will be proved elsewhere.

Throughout the paper the norm $\|\cdot\|_p$ ($1 \leq p \leq \infty$) in \mathbb{C}^n is defined as usual, i.e. $\|z\|_p = (\sum_{i=1}^n |z_i|^p)^{1/p}$. For a given point z in \mathbb{C}^n we define

$$(7) \quad d(z) = (d_1(z), \dots, d_n(z)) \quad \text{and} \quad \delta(z) = \min\{d_1(z), \dots, d_n(z)\},$$

where $d_i(z) = \min\{|z_i - z_j| : j \neq i\}$ for $i = 1, 2, \dots, n$. For a given point z in \mathbb{C}^n with distinct components we use the notations

$$(8) \quad \frac{W(z)}{d(z)} = \left(\frac{W_1(z)}{d_1(z)}, \dots, \frac{W_n(z)}{d_n(z)} \right) \quad \text{and} \quad E(z) = \left\| \frac{W(z)}{d(z)} \right\|_p.$$

2 Localization of simple polynomial zeros

The following theorem is an improvement of a result of ZHENG [17].

Theorem 2.1. *Let f be a monic polynomial of degree $n \geq 2$. Suppose there exist $1 \leq p \leq \infty$ and a point z^0 in \mathbb{C}^n with distinct components such that*

$$(9) \quad E(z^0) < 1/2^{1/q} \quad \text{and} \quad \phi(E(z^0)) < 1.$$

where q is defined by $1/p + 1/q = 1$ and ϕ is a real function defined on $[0, 1/2^{1/q})$ by

$$(10) \quad \phi(x) = \frac{(n-1)^{1/q}x}{(1-x)(1-2^{1/q}x)} \left(1 + \frac{x}{(n-1)^{1/p}(1-2^{1/q}x)} \right)^{n-1}.$$

Then the following statements hold true.

(i) *The polynomial f has only simple zeros.*

(ii) *The closed disks*

$$D_i = \{z \in \mathbb{C} : |z - (z_i^0 - W_i(z^0))| \leq C |W_i(z^0)|\}, \quad i = 1, 2, \dots, n,$$

where $C = \theta\lambda/(1 - \theta\lambda^2)$, $\lambda = \phi(E(z^0))$, $\theta = 1 - 2^{1/q}E(z^0)$, are mutually disjoint and each of them contains exactly one zero of f .

Note that under the conditions (9) Weierstrass' method (1) is convergent with the second order of convergence (see PROINOV [18]).

3 Semilocal convergence of Ehrlich's method

Theorem 3.1. *Let f be a monic polynomial of degree $n \geq 2$, $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. Define the real function*

$$(11) \quad \phi(x) = \frac{ax^2}{(1 - (a+1)x)(1 - (a+b)x)} \left(1 + \frac{1}{n-1} \cdot \frac{ax}{1 - (a+b)x} \right)^{n-1},$$

where $a = (n-1)^{1/q}$ and $b = 2^{1/q}$. Suppose that z^0 is an initial point in \mathbb{C}^n with distinct components satisfying

$$E(z^0) < 1/(a+b) \quad \text{and} \quad \phi(E(z^0)) \leq 1.$$

Then the following statements hold true.

- (i) *Ehrlich's iterative sequence (2) is well-defined and convergent to a root-vector ξ of f . Moreover, if $\phi(E(z^0)) < 1$, then the order of convergence is three.*
- (ii) *For each $k \geq 1$ we have the following a priori error estimate*

$$(12) \quad \|z^k - \xi\|_p \leq A_k \frac{\theta^k \lambda^{(3^k-1)/2}}{1 - \theta \lambda^{3^k}} \|W(z^0)\|_p,$$

where $\lambda = \phi(E(z^0))$, $\theta = \psi(E(z^0))$, $A_k = \mu(E(z^0) \lambda^{(3^k-1)/2})$ and the real functions ψ and μ are defined by

$$\psi(x) = \frac{1 - (a+b)x}{1 - ax} \quad \text{and} \quad \mu(x) = \frac{1}{1 - ax}.$$

- (iii) *For all $k \geq 0$ we have the following a posteriori error estimate*

$$(13) \quad \|z^k - \xi\|_p \leq \frac{\mu_k}{1 - \theta_k \lambda_k} \|W(z^k)\|_p,$$

where $\lambda_k = \phi(E(z^k))$, $\theta_k = \psi(E(z^k))$ and $\mu_k = \mu(E(z^k))$.

Setting $p = \infty$ in Theorem 3.1 we obtain the following corollary.

Corollary 3.1. *Let f be a monic polynomial of degree $n \geq 2$ and let C be a real number satisfying*

$$(14) \quad 0 \leq C < \frac{1}{n+1} \quad \text{and} \quad \frac{(n-1)C^2}{(1-nC)(1-(n+1)C)} \left(\frac{1-nC}{1-(n+1)C} \right)^{n-1} < 1.$$

Suppose that z^0 is an initial point in \mathbb{C}^n satisfying

$$(15) \quad \left\| \frac{W(z^0)}{d(z^0)} \right\|_\infty \leq C.$$

Then Ehrlich's method (2) is convergent to a root-vector ξ of f with the third order of convergence. Moreover, the error bounds (12) and (13) hold with $p = \infty$.

Corollary 3.1 improves Theorem 4.1 of PETKOVIĆ and HERCEG [15]. Note that Petkovic and Herceg have proved only linear convergence of Ehrlich's method under the stronger condition $\|W(z^0)\|_\infty \leq C \delta(z^0)$ with C satisfying (14) as well as

$$(16) \quad \beta := \frac{(n-1)C^2(1+(n-1)C)}{(1-nC)(1-(n-1)C)} \left(\frac{1-nC}{1-(n+1)C} \right)^{n-1} < 1 \quad \text{and}$$

$$(17) \quad g(\beta) < \frac{1 - (n-1)C}{2C} \quad \text{where} \quad g(x) = \begin{cases} 1 + 2x & \text{for } 0 < x \leq 1/2; \\ 1/(1-x) & \text{for } 1/2 < x < 1. \end{cases}$$

Corollary 3.1 shows that Petkovic-Herceg's assumptions (16) and (17) can be omitted.

The following result improves and generalizes Theorem 1 of ZHENG and HUANG [14] as well as previous results [9, 10, 11, 12].

Corollary 3.2. *Let f be a monic polynomial of degree $n \geq 3$, $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. Suppose z^0 is an initial point in \mathbb{C}^n satisfying*

$$(18) \quad \left\| \frac{W(z^0)}{d(z^0)} \right\|_p \leq \frac{1}{2(n-1)^{1/q} + 2}.$$

Then f has only simple zeros and Ehrlich's method (2) converges to a root-vector ξ of f with the third order of convergence. Moreover, the error estimates (12) and (13) hold.

Proof. From Theorem 2.1 and the proof of Corollary 2 of [18] we conclude that f has only simple zeros. It is easy to compute that

$$\phi(R) = \frac{a}{(a+1)(a+2-b)} \left(1 + \frac{1}{n-1} \cdot \frac{a}{a+2-b} \right)^{n-1}.$$

for $R = 1/(2a+2)$, where a , b and ϕ are defined as in Theorem 3.1. By Theorem 3.1 it suffices to prove that $\phi(R) \leq 1$. If $a > e-1$, then $\phi(R) < e/(a+1) < 1$. If $a \leq e-1$, then $\phi(R) < (e-1)/(a-b+2) \leq (e-1)/2 < 1$ which completes the proof. \square

Remark. In Corollary 3.2 we consider an initial condition of the type

$$E(z^0) \leq \frac{1}{A(n-1)^{1/q} + B}.$$

Note that such initial conditions can be obtained for every $A > 1$. For example, if $A > 1$, then one can take

$$B = 2^{1/q} + \frac{1}{4(A-1)} \exp \frac{1}{A-1}.$$

Corollary 3.3. *Let f be a monic polynomial of degree $n \geq 2$. Suppose that z^0 is an initial point in \mathbb{C}^n satisfying*

$$\left\| \frac{W(z^0)}{d(z^0)} \right\|_\infty \leq \frac{1}{1.5n + 1.8}.$$

Then Ehrlich's method (2) is convergent to a root-vector ξ of f with the third order of convergence. Moreover, the error estimates (12) and (13) hold with $p = \infty$.

Proof. Define $\phi_n = \phi(1/(1.5n + 1.8))$, where ϕ is defined by (11) with $p = \infty$. The sequence (ϕ_n) is increasing for $2 \leq n \leq 10$ and decreasing for $n \geq 10$. Hence $\phi_n \leq \phi_{10} < 1$ for $n \geq 2$. Now the conclusion follows from Theorem 3.1. \square

Corollary 3.3 improves Theorem 4.2 of PETKOVIĆ and HERCEG [15]. Note that the authors of this work have proved that Ehrlich's method is convergent under the condition $\|W(z^0)\|_\infty \leq C(n) \delta(z^0)$, where

$$C(n) = \begin{cases} 1/(n + 4.5) & \text{for } n = 3, 4; \\ 1/(1.545n + 5) & \text{for } n \geq 5. \end{cases}$$

Corollary 3.4. *Let f be a monic polynomial of degree $n \geq 2$. Suppose that z^0 is an initial point in \mathbb{C}^n satisfying*

$$\left\| \frac{W(z^0)}{d(z^0)} \right\|_1 \leq R = 0.2922 \dots$$

where R is the unique positive solution of the equation

$$(19) \quad \left(\frac{x}{1 - 2x} \right)^2 \exp \frac{x}{1 - 2x} = 1.$$

Then f has only simple zeros and Ehrlich's method (2) converges to a root-vector ξ of f with the third order of convergence and the estimates (12) and (13) hold with $p = 1$.

Proof. It follows from Theorem 2.1 and the proof of Corollary 3 of [18] that f has only simple zeros. It is easy to show that $\phi(x) < g(x)$ for $0 < x < 1/2$, where $\phi(x)$ is defined by (11) with $p = 1$ and $g(x)$ denotes the left-hand side of the equation (19). Therefore, $\phi(R) < 1$ which according to Theorem 3.1 completes the proof. \square

4 Semilocal convergence of Nourein's method

Theorem 4.1. *Let f be a monic polynomial of degree $n \geq 2$, $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. Define the real function*

$$(20) \quad \phi(x) = \frac{a^2 x^3}{(1 - mx + bx^2)(1 - (a + 2)x + x^2)} \left(1 + \frac{1}{n - 1} \cdot \frac{a(x - x^2)}{1 - mx + bx^2} \right)^{n-1},$$

where $a = (n - 1)^{1/q}$, $b = 2^{1/q}$ and $m = a + b + 1$. Suppose that z^0 is an initial point in \mathbb{C}^n with distinct components satisfying

$$E(z^0) < 2/(m + \sqrt{m^2 - 4b}) \quad \text{and} \quad \phi(E(z^0)) \leq 1.$$

Then the following statements hold true.

- (i) Nourein's iterative sequence (5) is well-defined and convergent to a root-vector ξ of f . Moreover, if $\phi(E(z^0)) < 1$, then the order of convergence is four.
- (ii) For each $k \geq 1$ we have the following a priori error estimate

$$(21) \quad \|z^k - \xi\|_p \leq \mu_k \frac{\theta^k \lambda^{(4^k-1)/3}}{1 - \theta \lambda^{4^k}} \|W(z^0)\|_p,$$

where $\lambda = \phi(E(z^0))$, $\theta = \psi(E(z^0))$, $\mu_k = \mu(E(z^0) \lambda^{(4^k-1)/3})$ and the real functions ψ and μ are defined by

$$\psi(x) = \frac{1 - mx + bx^2}{1 - (a+1)x} \quad \text{and} \quad \mu(x) = \frac{1 - x}{1 - (a+1)x}.$$

- (iii) For all $k \geq 0$ we have the following a posteriori error estimate

$$(22) \quad \|z^k - \xi\|_p \leq \frac{\mu_k}{1 - \theta_k \lambda_k} \|W(z^k)\|_p,$$

where $\lambda_k = \phi(E(z^k))$, $\theta_k = \psi(E(z^k))$ and $\mu_k = \mu(E(z^k))$.

Setting $p = \infty$ in Theorem 4.1 we obtain the following corollary.

Corollary 4.1. *Let f be a monic polynomial of degree $n \geq 2$ and let C be a real number such that $0 \leq C < 2/(n+2+\sqrt{n^2+4n-8})$ and*

$$(23) \quad \frac{(n-1)^2 C^3}{(1 - (n+1)C + C^2)(1 - (n+2)C + 2C^2)} \left(\frac{1 - (n+1)C + C^2}{1 - (n+2)C + 2C^2} \right)^{n-1} < 1.$$

Suppose that z^0 is an initial point in \mathbb{C}^n satisfying (15). Then Nourein's method (5) is convergent to a root-vector ξ of f with the fourth order of convergence. Moreover, the error bounds (21) and (22) hold with $p = \infty$.

Corollary 4.1 improves Theorem 2 of NEDIĆ [16]. Nedić has proved only linear convergence of Nourein's method under the condition $\|W(z^0)\|_\infty \leq C \delta(z^0)$ with C satisfying

$$0 < C < \frac{2}{n+4+\sqrt{n^2+8n}} \quad \text{and} \quad \beta < \frac{1 - nC}{1 + (n-2)C},$$

where β denotes the left-hand side of (23).

The following corollary improves and generalizes Theorem 2 of ZHENG and HUANG [14] as well as previous results [10, 12, 13].

Corollary 4.2. *Under the assumptions of Corollary 3.2 f has only simple zeros and Nouredin's method (5) converges to a root-vector ξ of f with the fourth order of convergence. Moreover, we have the error estimates (21) and (22).*

Proof. Corollary 3.2 implies that f has only simple zeros. It is easy to compute that

$$\phi(R) = \frac{2a^2(a+1)}{(2(a+1)^2 - (2a+1)b)(2a^2 + 2a + 1)} \left(1 + \frac{1}{n-1} \cdot \frac{a(2a+1)}{2(a+1)^2 - (2a+1)b} \right)^{n-1}$$

for $R = 1/(2a+2)$, where a , b and ϕ are defined as in Theorem 4.1. Taking into account that $b \leq 2$ we get

$$(24) \quad \phi(R) \leq \frac{2a^2(a+1)}{(2(a+1)^2 - (2a+1)b)(2a^2 + 2a + 1)} \left(1 + \frac{1}{n-1} \cdot \frac{2a+1}{2a} \right)^{n-1}$$

and

$$(25) \quad \phi(R) \leq \frac{a+1}{2a^2 + 2a + 1} \left(1 + \frac{1}{n-1} \cdot \frac{2a+1}{2a} \right)^{n-1}.$$

By Theorem 4.1 it suffices to prove that $\phi(R) \leq 1$. We shall consider two cases.

Case 1. Suppose $a \geq 1.8$. It follows from (25) that

$$(26) \quad \phi(R) < g(a) = \frac{a+1}{2a^2 + 2a + 1} \exp \left(1 + \frac{1}{2a} \right).$$

The function g is decreasing on $(0, \infty)$. Therefore, $\phi(R) < g(a) < g(1.8) < 1$.

Case 2. Suppose $a \leq 1.8$. From (24) and the obvious inequality $b \leq a$ we obtain

$$(27) \quad \phi(R) < h(a) = \frac{2a^2(a+1)}{(3a+2)(2a^2 + 2a + 1)} \exp \frac{3}{2}.$$

The function h is increasing on $(0, \infty)$. Therefore, $\phi(R) < h(a) < h(1.8) < 1$ which completes the proof. □

Corollary 4.3. *Let f be a monic polynomial of degree $n \geq 2$. Suppose that z^0 is an initial point in \mathbb{C}^n satisfying*

$$\left\| \frac{W(z^0)}{d(z^0)} \right\|_{\infty} \leq \frac{1}{1.4n + 2.8}.$$

Then Nouredin's method (5) is convergent to a root-vector ξ of f with the fourth order of convergence. Moreover, the estimates (21) and (22) hold with $p = \infty$.

Proof. Define $\phi_n = \phi(1/(1.4n + 2.8))$, where ϕ is defined by (20) with $p = \infty$. The sequence (ϕ_n) is increasing for $2 \leq n \leq 19$ and decreasing for $n \geq 19$. Hence $\phi_n \leq \phi_{19} < 1$ for $n \geq 2$. Now the conclusion follows from Theorem 4.1. \square

Corollary 4.3 improves Theorem 3 of NEDIĆ [16]. Note that Nedić has proved that Nourein's method is convergent under the condition $\|W(z^0)\|_\infty \leq C(n)\delta(z^0)$ where

$$C(n) = \begin{cases} 1/(1.64n + 1.944) & \text{for } 3 \leq n \leq 23; \\ 1/(1.42n + 8.7) & \text{for } n > 23. \end{cases}$$

Corollary 4.4. *Let f be a monic polynomial of degree $n \geq 2$. Suppose that z^0 is an initial point in \mathbb{C}^n satisfying*

$$\left\| \frac{W(z^0)}{d(z^0)} \right\|_1 \leq R = 0.2825 \dots,$$

where R is the unique solution of the equation

$$(28) \quad \frac{x^3}{(1 - 3x + x^2)^2} \exp \frac{x - x^2}{1 - 3x + x^2} = 1$$

in the interval $(0, 2/(3 + \sqrt{5}))$. Then f has only simple zeros and Nourein's sequence (5) converges to a root-vector ξ of f with the fourth order of convergence and error bounds (21) and (22) with $p = 1$.

Proof. By Theorem 3.4 f has only simple zeros. It is easy to show that $\phi(x) < g(x)$ for $0 < x < 2/(3 + \sqrt{5})$, where $g(x)$ denotes the left-hand side of the equation (28) and $\phi(x)$ is defined by (20) with $p = 1$. Therefore, $\phi(R) < 1$ which according to Theorem 4.1 completes the proof. \square

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